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Wigner distributions for finite-state systems without redundant phase-point operators

S Chaturvedi¹, N Mukunda² and R Simon³

¹ School of Physics, University of Hyderabad, Hyderabad 500046, India

² Centre for High Energy Physics, Indian Institute of Science, Bangalore 560012, India

³ The Institute of Mathematical Sciences, C. I. T. Campus, Chennai 600113, India

E-mail: scsp@uohyd.ernet.in, nmukunda@cts.iisc.ernet.in and simon@imsc.res.in

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Abstract

We set up Wigner distributions for N -state quantum systems following a Dirac-inspired approach. In contrast to much of the work in this study, requiring a $2N \times 2N$ phase space, particularly when N is even, our approach is uniformly based on an $N \times N$ phase-space grid and thereby avoids the necessity of having to invoke a ‘quadrupled’ phase space and hence the attendant redundance. Both N odd and even cases are analysed in detail and it is found that there are striking differences between the two. While the N odd case permits full implementation of the marginal property, the even case does so only in a restricted sense. This has the consequence that in the even case one is led to several equally good definitions of the Wigner distributions as opposed to the odd case where the choice turns out to be unique.

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1. Introduction

The Wigner distribution formalism [1–3], as a way of describing general quantum mechanical states, was originally defined for systems with continuous Cartesian configuration spaces. For a quantum system with one Cartesian coordinate, this distribution for the pure state case is

$$W_{\psi}(q, p) = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} dq' \psi(q - q'/2) \psi^*(q + q'/2) e^{ipq'/\hbar}, \quad (1)$$

where $\psi(q)$ is the complex Schrödinger wavefunction. (The generalization to m Cartesian coordinates is straightforward.) This is a real function on the continuous and Euclidean (more correctly linear symplectic) classical phase space. This permits a comparison to classical statistical mechanics. A general Wigner distribution differs from a genuine probability distribution in two important respects: (a) it can become negative in parts of the phase

space, and (b) it is bounded above in magnitude by $1/\pi\hbar$. It cannot, therefore, be interpreted as a probability distribution and for this reason it is referred to as a phase-space quasi-probability distribution. Nevertheless, upon integrating over the momenta (or the positions), the Wigner distribution does yield the configuration space (or momentum space) probability distributions specified by quantum mechanics. This is referred to as the recovery of the standard marginals. By exploiting the action of the linear symplectic group on the classical phase space, this property generalizes i.e. integrating the Wigner distribution over half the phase-space variables corresponding to a so-called Lagrangian subspace always results in a genuine probability distribution with a clear quantum mechanical meaning.

For the Cartesian case, it has been shown [2] that the original Wigner distribution is the only one that possesses several natural covariance and other properties that one can demand of a phase-space quasi-probability distribution.

The Wigner distribution has been generalized to quantum systems with different types of ‘configuration spaces’ [4–30]. One can consider both continuous and discrete possibilities here. A continuous non-Cartesian instance is when the configuration space is the manifold of a semi-simple compact Lie group [23]. Extension of the notion of a Wigner distribution to such non-Cartesian situations becomes possible by rewriting (1) in a more suggestive form as

$$W_\psi(q, p) = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} dq' \int_{-\infty}^{\infty} dq'' \delta(q - (q' + q'')/2) \psi^*(q') \psi(q'') e^{ipq'/\hbar} e^{-ipq''/\hbar}, \quad (2)$$

which reveals the crucial role of the concept of the midpoint $(q' + q'')/2$ of two points q' and q'' in the Wigner construction. Thus, given a complex Schrödinger wavefunction on the group, one can construct a Wigner distribution as a partial integral of the product of the wavefunction and its conjugate at independent points, with a kernel involving an appropriate generalization of the ‘midpoint rule’ above. The arguments of the Wigner distribution are determined in an interesting way, and the standard marginal property is recovered [23].

Three other major directions in which generalizations have been attempted are discrete finite-dimensional quantum systems with states corresponding to finite-dimensional complex Hilbert spaces. (Hereafter, dimension of a quantum system will be taken to mean the dimension of the underlying complex Hilbert space.) The situations where the coordinates take values in

- a finite field \mathbb{F}_{p^n} ,
- a ring \mathbb{Z}_N ,
- a finite group (Abelian or non-Abelian)

have been studied. The first two are of particular interest owing to potential applications to quantum information processing and state determination.

Typically, odd-dimensional quantum systems have properties distinctly different from even-dimensional ones, at least as far as Wigner distributions are concerned. For instance, in the third case listed above, for odd dimensions an adaptation of the midpoint rule alluded to above leads almost uniquely to a very satisfactory definition of Wigner distributions [24, 25]. We use the qualification ‘almost’ here because, as it turns out, every way of viewing a set of odd N objects as a group of order N leads to a corresponding way of defining Wigner distributions for an N -dimensional quantum state. The cyclic group of order N is always available, but for specific N other non-Abelian possibilities may exist. The smallest odd value of N for which this happens is $N = 21$ [24]. The midpoint rule, however, always works.

The case when coordinates take values in a finite field has been well analysed [7, 11]. The field structure ensures that the resultant phase space, albeit discrete, inherits many geometrical features of the Cartesian case. Thus, one has a well-defined notion of an isotropic line

(analogue of a straight line through the origin in the Cartesian case) and the notion of lines parallel to it giving rise to a ‘striation’ (discrete analogue of foliation) of the phase space. Furthermore, except at the origin, each phase point lies on exactly one isotropic line and any two non-parallel lines intersect at exactly one point. The work of Wootters *et al* [7] elegantly exploits these geometrical features of the phase space to arrive at the concept of a quantum net whereby one associates a rank-1 projector with each line in the phase space in a manner that is consistent with translational covariance. The requirement of translational covariance has the consequence that the projectors associated with lines in a striation must be trace orthogonal and that the projectors associated with lines in different striations must be mutually unbiased. Each quantum net set up in this manner then defines a Wigner distribution which, by construction, respects the marginal property along all isotropic lines.

In the present work we focus our attention on the second case listed above and construct the phase space and Wigner distributions thereon using what we call a Dirac-inspired approach to the problem [26]. This approach was originally presented in the continuous Cartesian case and essentially requires computing the square root of a certain phase-space kernel. In that case, thanks to the continuity of the phase space, this problem has a unique answer and one recovers the well-known Wigner distribution. This approach, however, can be extended to the finite dimensions but then the discreteness brings in undetermined signs, one at each phase point. The imposition of the standard marginal properties determines only some of these signs. We then examine to what extent these signs can be further fixed or related to one another if one imposes further marginal conditions. This is analogous to exploiting general Lagrangian subspaces in the Cartesian case. Thus, we demand that the phase-space averages of the Wigner distribution along general ‘isotropic lines’ and lines ‘parallel’ to it yield bonafide probability distributions. Unlike finite fields, not all non-zero elements of \mathbb{Z}_N possess multiplicative inverses. The description and determination of these lines therefore involve in depth number-theoretic niceties and have been recently given [31]. For the odd N case one can consistently impose marginal property on all isotropic lines and thus uniquely fix all the signs.

For even N , however, this is not so. We find that the marginal property cannot be consistently imposed on all isotropic lines but only on specific subsets of them (i.e. on particular orbits under the action of the discrete symplectic group $SL(2, \mathbb{Z}_N)$). Therefore, the best one can do is to demand the marginal property of the largest such subset. This is actually good enough as the isotropic lines in this subset (and only in this subset) cover all phase points.

We emphasize in passing that for odd N the action and orbit structure under $SL(2, \mathbb{Z}_N)$ in phase space are not needed in solving our problem, but for even N they turn out to be essential.

We explicitly carry out this programme for even N and find that, unlike the odd case, all signs do not get fixed and therefore one has a family of Wigner distributions characterized by different choices of signs, all of which respect the marginal property. We emphasize that here we work all along with an $N \times N$ phase-space lattice and the associated N^2 phase-point operators. This is in contrast to other formulations in the literature where a $2N \times 2N$ phase-space lattice is invoked to arrive at a satisfactory definition of Wigner distributions for an N -state quantum system. The recent work of Bar-on [30], based on symmetric informationally complete projection operator valued measures (SIC-POVMs) [32] and a block design theory [33] inspired picture of phase space, reduces the number of phase points from $4N^2$ to $N^2 + N$, but this still exceeds by N the number of phase points required in the present work.

A brief outline of this work is as follows. In section 2 we set up our notation and summarize the properties of the operators used later. In section 3 we briefly recapitulate the Dirac-inspired approach to Wigner distributions and give the expressions for the phase-point operators in a convenient form along with the conditions on the undetermined signs that appear therein in order that the phase-point operators satisfy the standard marginal property.

In section 4 we briefly summarize the properties of the isotropic lines in our phase space as made available in a recent work by Albouy [31]. In section 5 we discuss the odd case and in section 6 summarize our result for the case when N is a prime power. The case of general N is discussed in section 7. In section 8 we discuss the question of tomographic reconstruction for the case when $N = 2^n$. We conclude with section 9 where we note that although for a given even N there is a great variety of Wigner distributions characterized by different choices for the unfixed signs, the actual number of choices available, in so far as the eigenvalues of the relevant phase-point operators are concerned, is rather small. Finally, in a short appendix, we give, for arbitrary N , the matrix elements, in the coordinate basis, for all operators that we introduce in the present work and, by way of illustration, explicitly display the matrices for the special case corresponding to $N = 2$.

2. Preliminaries

Consider a quantum system described by a complex Hilbert space \mathcal{H} of dimension N . Denote by $\{|q\rangle\}, \{|p\rangle\}$, $q, p \in \mathbb{Z}_N$, two orthonormal bases—coordinate and momentum bases—related to each other by a finite Fourier transform:

$$\langle q'|q\rangle = \delta_{q',q}; \quad \langle p'|p\rangle = \delta_{p',p}; \quad \langle q|p\rangle = \omega^{qp}/\sqrt{N}; \quad \omega = e^{2\pi i/N}. \quad (3)$$

Here, \mathbb{Z}_N stands for the ring of integers $\{0, 1, 2, \dots, N - 1\}$ with addition and multiplication modulo N . We use the notation $[n]$ to denote n modulo N . On the Hilbert space \mathcal{H} we introduce the familiar Weyl operators:

$$U = \omega^{\hat{q}} = \sum_{q \in \mathbb{Z}_N} \omega^q |q\rangle\langle q|, \quad U|p\rangle = |[p + 1]\rangle; \quad U^\dagger U = U^N = 1, \quad (4)$$

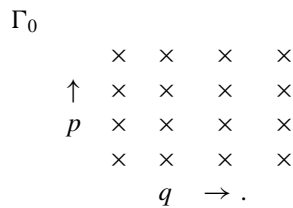
$$U^p = \omega^{p\hat{q}} = \sum_{q \in \mathbb{Z}_N} \omega^{pq} |q\rangle\langle q|; \quad |q\rangle\langle q| = \frac{1}{N} \sum_{p \in \mathbb{Z}_N} \omega^{-qp} U^p,$$

$$V = \omega^{-\hat{p}} = \sum_{p \in \mathbb{Z}_N} \omega^{-p} |p\rangle\langle p|, \quad V|q\rangle = |[q + 1]\rangle; \quad V^\dagger V = V^N = 1,$$

$$V^q = \omega^{-q\hat{p}} = \sum_{p \in \mathbb{Z}_N} \omega^{-pq} |p\rangle\langle p|; \quad |p\rangle\langle p| = \frac{1}{N} \sum_{q \in \mathbb{Z}_N} \omega^{qp} V^q, \quad (5)$$

$$U^p V^q = \omega^{pq} V^q U^p = \tau^{2pq} V^q U^p,$$

where $\tau = e^{\pi i/N}$. We denote by Γ_0 the discrete ‘classical’ phase space of N^2 points $\sigma = (q, p)$ equipped with the symplectic product $\langle \sigma, \sigma' \rangle = pq' - qp'$ between any two phase points σ and σ' . It is helpful to arrange the values of q and p in the usual Cartesian fashion and to picture Γ_0 as an $N \times N$ grid as shown below for $N = 4$:



We denote by \mathcal{K} the Hilbert space of square summable functions on Γ_0 :

$$\begin{aligned} \mathcal{K} &= L^2(\Gamma_0) = N^2\text{-dimensional Hilbert space} \\ &= \left\{ f(\sigma) \in C \mid \sigma \in \Gamma_0, \|f\|^2 = \sum_{\sigma \in \Gamma_0} |f(\sigma)|^2 \right\}. \end{aligned} \quad (6)$$

Γ_0 can be viewed as an Abelian group of order N^2 :

$$\begin{aligned} \sigma \in \Gamma_0 &\rightarrow \text{group element } g(\sigma) = a^q b^p; \\ \text{generators } a, b &: ab = ba, a^N = b^N = e; \\ g((0, 0)) &= \text{identity } e; \end{aligned} \tag{7}$$

$$\begin{aligned} \text{composition: } g(\sigma')g(\sigma) &= g([q' + q], [p' + p]) = g([\sigma + \sigma']); \\ \text{inversion: } g(\sigma)^{-1} &= g([N - q], [N - p]) \equiv g([N - \sigma]). \end{aligned} \tag{8}$$

Thus, Γ_0 is the direct product $G_0 \times G_0$ where G_0 is the (Abelian) cyclic group of order N .

Γ_0 has N^2 distinct inequivalent one-dimensional unitary irreducible representations (UIRs), with characters labelled by points $\sigma' \in \Gamma_0$:

$$\begin{aligned} \sigma' \in \Gamma_0: \quad \text{UIR } a &\rightarrow \omega^{-p'}, b \rightarrow \omega^{q'}; \\ g(\sigma) &\rightarrow \chi_{\sigma'}(\sigma) = \omega^{pq' - qp'} = \omega^{(\sigma, \sigma')}; \\ \text{orthogonality: } \chi_{\sigma''}^\dagger \chi_{\sigma'} &= \sum_{\sigma \in \Gamma_0} \chi_{\sigma''}^*(\sigma) \chi_{\sigma'}(\sigma) \\ &= \sum_{q, p \in \mathbb{Z}_N} \omega^{p''q - q''p + q'p - p'q} = N^2 \delta_{\sigma'', \sigma'}. \end{aligned} \tag{9}$$

The set of functions $\{\frac{1}{N} \chi_{\sigma'}(\sigma)\}$ on Γ_0 forms an orthonormal basis (ONB) for \mathcal{K} . The trivial UIR is $\sigma' = 0$, $\chi_0(\sigma) = 1$, and hence

$$\sum_{\sigma \in \Gamma_0} \chi_{\sigma''}(\sigma) = \sum_{\sigma \in \Gamma_0} \omega^{(\sigma, \sigma'')} = N^2 \delta_{\sigma'', 0}. \tag{10}$$

Finally, we introduce the displacement operators $D(\sigma)$ on \mathcal{H} , one for each phase point, and summarize their properties:

$$\sigma \in \Gamma_0: \quad D(\sigma) \equiv D(q, p) = \tau^{qp} V^q U^p = \tau^{-qp} U^p V^q; \tag{11}$$

$$\text{unitarity: } D(\sigma)^\dagger D(\sigma) = 1 \text{ on } \mathcal{H}; \tag{12}$$

$$\text{inverses: } D(\sigma)^{-1} = \eta_\sigma D([N - \sigma]); \tag{13}$$

$$\begin{aligned} \eta_\sigma &= \begin{cases} 1 & \text{for } q = 0 \text{ or } p = 0 \\ (-1)^{q+p+N} & \text{for } 1 \leq q, p \leq N - 1 \end{cases} \\ &= \tau^{-[N-q][N-p]+qp}; \end{aligned} \tag{14}$$

$$\text{composition: } D(\sigma')D(\sigma) = \tau^{(\sigma', \sigma)} \epsilon(\sigma', \sigma) D([\sigma' + \sigma]); \tag{15}$$

$$\begin{aligned} \epsilon(\sigma', \sigma) = \epsilon(\sigma, \sigma') &= \begin{cases} 1 & \text{for } \sigma' + \sigma \in \Gamma_0 \\ (-1)^{q'+q} & \text{for } q' + q \leq N - 1, p' + p \geq N \\ (-1)^{p'+p} & \text{for } q' + q \geq N, p' + p \leq N - 1 \\ (-1)^{q'+q+p'+p+N} & \text{for } q' + q \geq N, p' + p \geq N \end{cases} \\ &= \tau^{(q'+q)(p'+p) - [q'+q][p'+p]}; \end{aligned} \tag{16}$$

$$\text{trace orthogonality: } \text{Tr}(D(\sigma')^\dagger D(\sigma)) = N \delta_{\sigma', \sigma}. \tag{17}$$

The set of operators $\{\frac{1}{\sqrt{N}}D(\sigma)\}$ constitutes a complete irreducible trace orthonormal set of operators on \mathcal{H} satisfying the relations

$$D(\sigma')D(\sigma) = \omega^{(\sigma',\sigma)}D(\sigma)D(\sigma'); \tag{18}$$

$$D(\sigma')D(\sigma)D(\sigma')^{-1} = \omega^{(\sigma',\sigma)}D(\sigma). \tag{19}$$

Thus, $\{D(\sigma)\}$ form an irreducible unitary N -dimensional ray representation of Γ_0 on \mathcal{H} . Some useful relations are given below:

$$\sigma \in \Gamma_0: \quad \tau^{-qP}D(\sigma) = V^qU^P = \sum_{\sigma' \in \Gamma_0} \omega^{(\sigma,\sigma')}|p'\rangle\langle q'|, \tag{20}$$

$$|p\rangle\langle q| = \frac{1}{N^2} \sum_{\sigma' \in \Gamma_0} \omega^{(\sigma,\sigma')} \tau^{-q'P'} D(\sigma'). \tag{21}$$

3. Dirac-inspired approach to Wigner distributions

The central idea in this approach, discussed in detail in [26], is to initially associate with any operator \hat{A} a phase-space function $A(\sigma)$ constructed out of its mixed matrix elements, such as $\langle q|\hat{A}|p\rangle$, in such a way that the trace of the product of two operators is expressed as a phase-space average of a kernel times the product of their phase-space functions, and then try to transform away this kernel. As shown in [26], this exercise entails finding a symmetric square root of the kernel $K(\sigma, \sigma')$ defined below:

$$K(\sigma; \sigma') = K(\sigma'; \sigma) = \omega^{(q-q')(p-p')}. \tag{22}$$

3.1. The kernel K and its symmetric square roots ξ

The kernel K defines an operator on \mathcal{K} . The characters $\{\chi_{\sigma'}(\sigma)\}$ of Γ_0 form a complete orthogonal set of eigenvectors of K in \mathcal{K} :

$$\sum_{\sigma' \in \Gamma_0} K(\sigma; \sigma')\chi_{\sigma'}(\sigma') = N\omega^{q''p''}\chi_{\sigma''}(\sigma). \tag{23}$$

Using the results above for the eigenvalues and eigenvectors of K , a general square root ξ of K is defined by

$$\xi\chi_{\sigma''} = \sqrt{N}\tau^{q''p''}S(\sigma'')\chi_{\sigma''}, \quad S(\sigma'') = \pm 1, \quad \sigma'' \in \Gamma_0. \tag{24}$$

At this point, there are N^2 sign choices. This freedom will be reduced as we proceed. The kernel $\xi(\sigma; \sigma')$ is

$$\begin{aligned} \xi(\sigma; \sigma') &= \frac{1}{N^2}\sqrt{N} \sum_{\sigma'' \in \Gamma_0} \tau^{q''p''}S(\sigma'')\chi_{\sigma''}(\sigma)\chi_{\sigma''}(\sigma')^* \\ &= \frac{1}{N^{3/2}} \sum_{\sigma'' \in \Gamma_0} \tau^{q''p''}S(\sigma'')\omega^{(\sigma,\sigma'')-(\sigma',\sigma'')}. \end{aligned} \tag{25}$$

$K(\sigma; \sigma')$ is symmetric under $\sigma \leftrightarrow \sigma'$. Demanding the same for $\xi(\sigma; \sigma')$ places conditions on the signs $S(\sigma'')$. Since in general

$$\sum_{q \in \mathbb{Z}_N} f(q) = \sum_{q \in \mathbb{Z}_N} f([N - q]) \quad \text{etc,} \tag{26}$$

$$\begin{aligned}
 \xi(\sigma; \sigma') = \xi(\sigma'; \sigma) &\iff \sum_{\sigma'' \in \Gamma_0} \tau^{q'' p''} S(\sigma'') \omega^{(\sigma, \sigma'') - (\sigma', \sigma'')} \\
 &= \sum_{\sigma'' \in \Gamma_0} \tau^{q'' p''} S(\sigma'') \omega^{(\sigma', \sigma'') - (\sigma, \sigma'')} \\
 &= \sum_{\sigma'' \in \Gamma_0} \tau^{[N-q''] [N-p'']} S([N - \sigma'']) \omega^{(\sigma, \sigma'') - (\sigma', \sigma'')} \\
 &\iff S(\sigma) = \tau^{[N-q] [N-p] - qp} S([N - \sigma]) = \eta_\sigma S([N - \sigma]). \tag{27}
 \end{aligned}$$

So the number of independent $S(\sigma)$'s is about halved. We hereafter assume that $\xi(\sigma; \sigma')$ is symmetric.

3.2. Phase-point operators $\widehat{W}(\sigma)$

For any choice of the square root kernel $\xi(\sigma; \sigma')$, we define

$$\begin{aligned}
 \sigma \in \Gamma_0: \widehat{W}(\sigma) &= \sqrt{N} \sum_{\sigma' \in \Gamma_0} \xi(\sigma; \sigma') |p'\rangle \langle q'| \\
 &= \frac{1}{N} \sum_{\sigma' \in \Gamma_0} \omega^{(\sigma, \sigma')} S(\sigma') D(\sigma'). \tag{28}
 \end{aligned}$$

Condition (27) on $S(\sigma)$ arising from the symmetry of ξ ensures hermiticity of $\widehat{W}(\sigma)$:

$$\begin{aligned}
 \widehat{W}(\sigma)^\dagger &= \frac{1}{N} \sum_{\sigma' \in \Gamma_0} \omega^{(\sigma', \sigma)} S(\sigma') D(\sigma')^{-1} \\
 &= \frac{1}{N} \sum_{\sigma' \in \Gamma_0} \omega^{(\sigma', \sigma)} S([N - \sigma']) D([N - \sigma']) \\
 &= \widehat{W}(\sigma). \tag{29}
 \end{aligned}$$

Trace orthogonality of $D(\sigma)$ leads to

$$\text{Tr}(\widehat{W}(\sigma') \widehat{W}(\sigma)) = N \delta_{\sigma', \sigma}. \tag{30}$$

Further $\text{Tr}(D(\sigma)) = N \delta_{\sigma, 0}$ gives

$$\text{Tr}(\widehat{W}(\sigma)) = 1. \tag{31}$$

So both $\{\frac{1}{\sqrt{N}} \widehat{W}(\sigma)\}$ and $\{\frac{1}{\sqrt{N}} D(\sigma)\}$ are trace orthonormal complete sets of operators on \mathcal{H} .

From the conjugation relations (19) for D 's we get

$$\begin{aligned}
 D(\sigma') \widehat{W}(\sigma) D(\sigma')^{-1} &= \frac{1}{N} \sum_{\sigma'' \in \Gamma_0} \omega^{(\sigma, \sigma'')} S(\sigma'') \omega^{(\sigma', \sigma'')} D(\sigma'') \\
 &= \widehat{W}([\sigma + \sigma']). \tag{32}
 \end{aligned}$$

Recovery of standard marginals fixes some $S(\sigma)$:

$$\begin{aligned}
 \frac{1}{N} \sum_{p \in \mathbb{Z}_N} \widehat{W}(q, p) &= \frac{1}{N^2} \sum_{q', p', p \in \mathbb{Z}_N} \omega^{pq' - qp'} S(q', p') D(q', p') \\
 &= \frac{1}{N} \sum_{p' \in \mathbb{Z}_N} \omega^{-qp'} S(0, p') U^{p'} \\
 &= |q\rangle \langle q| \iff S(0, p') = 1; \tag{33}
 \end{aligned}$$

$$\frac{1}{N} \sum_{q \in \mathbb{Z}_N} \widehat{W}(q, p) = |p\rangle\langle p| \iff S(q', 0) = 1. \tag{34}$$

These are consistent with (27). So at this stage

$$S(q, 0) = S(0, p) = 1; \quad S(\sigma) = \eta_\sigma S([N - \sigma]). \tag{35}$$

Some useful formulae are given below:

$$\begin{aligned} \text{Tr}(\widehat{W}(\sigma)D(\sigma')^\dagger) &= \omega^{(\sigma, \sigma')} S(\sigma'), \\ \text{Tr}(\widehat{W}(\sigma)D(\sigma')) &= \omega^{(\sigma', \sigma)} S(\sigma'), \\ S(\sigma)D(\sigma) &= \frac{1}{N} \sum_{\sigma' \in \Gamma_0} \omega^{(\sigma, \sigma')} W(\sigma'). \end{aligned} \tag{36}$$

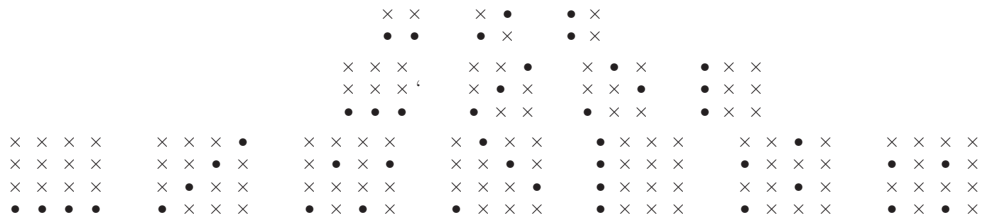
4. Isotropic lines and further marginals

The conditions so far on $S(\sigma)$ are given above in (35). To generate more conditions, we consider more marginal conditions, based on isotropic lines.

An isotropic line λ is a maximal set of N distinct points in Γ_0 including $\sigma = (0, 0)$ and obeying

$$\sigma', \sigma \in \lambda \Rightarrow \langle \sigma', \sigma \rangle = 0 \pmod N. \tag{37}$$

(The qualification ‘mod N ’ will frequently be left implicit.) Thus, for instance, for $N = 2, 3, 4$ one has 3, 4 and 7 isotropic lines, respectively. These are displayed below on their respective phase-space grids:



It is a fact that each point $\sigma \in \Gamma_0$ belongs to at least one isotropic line. We mention here some useful properties of such lines, and give further details in section 6. The maximality condition allows us to say

$$\sigma \in \Gamma_0, \quad \langle \sigma, \sigma' \rangle = 0 \quad \text{for all } \sigma' \in \lambda \Rightarrow \sigma \in \lambda. \tag{38}$$

This in turn leads to

$$\sigma \in \lambda \Rightarrow [N - \sigma] \in \lambda, \tag{39}$$

where $[N - \sigma]$ is defined in (8). We also have closure under the group composition law (7), (8) in Γ_0 :

$$\sigma', \sigma \in \lambda \Rightarrow [\pm 2\sigma], [\pm 3\sigma], \dots, [\sigma + \sigma'] \in \lambda. \tag{40}$$

In fact the points $\{\sigma\}$ of λ form an (Abelian) subgroup of Γ_0 , of order N , with group composition being (component-wise) addition mod N as in equations (7) and (8); the content is the same as in equation (40):

$$\sigma', \sigma \in \lambda \rightarrow g(\sigma')g(\sigma) = g([\sigma' + \sigma]), [\sigma' + \sigma] \in \lambda. \tag{41}$$

From this group structure we see that if any $\sigma, [\sigma' + \sigma] \in \lambda$ are given, then $\sigma' \in \lambda$ is uniquely determined. In case $g(\sigma)$ for $\sigma \in \lambda$ is an element of order N , λ itself is a cycle generated by σ and consisting of the N distinct points $\{(0, 0), \sigma, [2\sigma], [3\sigma], \dots, [(N - 1)\sigma]\}$. (However, a general λ need not be of this form.) Examples of such σ are $\sigma = (1, p)$ and $\sigma = (q, 1)$.

For any $\sigma' \in \Gamma_0$, we get a (one-dimensional) unitary irreducible representation (UIR) of λ by

$$\sigma \in \lambda \rightarrow \omega^{(\sigma, \sigma')}. \tag{42}$$

If $\sigma' \in \lambda$, this is the trivial UIR. If $\sigma' \notin \lambda$, $\langle \sigma, \sigma' \rangle \neq 0 \pmod N$ for some $\sigma \in \lambda$, so this is a nontrivial UIR. Hence, from orthogonality of inequivalent UIRs we obtain

$$\sum_{\sigma \in \lambda} \omega^{(\sigma, \sigma')} = N \text{ if } \sigma' \in \lambda, \quad 0 \text{ if } \sigma' \notin \lambda. \tag{43}$$

From relation (18) for $D(\sigma)$'s given earlier it follows that the operators $D(\sigma), \sigma \in \lambda$, form a mutually commuting set, but they may not form a representation of λ .

Isotropic line marginal condition. Given λ , define

$$\begin{aligned} P_\lambda &= \frac{1}{N} \sum_{\sigma \in \lambda} \widehat{W}(\sigma) = \frac{1}{N^2} \sum_{\sigma \in \lambda} \sum_{\sigma' \in \Gamma_0} \omega^{(\sigma, \sigma')} S(\sigma') D(\sigma') \\ &= \frac{1}{N} \sum_{\sigma \in \lambda} S(\sigma) D(\sigma). \end{aligned} \tag{44}$$

Clearly $P_\lambda^\dagger = P_\lambda$, $\text{Tr}(P_\lambda) = 1$. Now develop P_λ^2 :

$$\begin{aligned} P_\lambda^2 &= \frac{1}{N^2} \sum_{\sigma, \sigma' \in \lambda} S(\sigma) S(\sigma') D(\sigma) D(\sigma') \\ &= \frac{1}{N^2} \sum_{\sigma, \sigma' \in \lambda} S(\sigma) S(\sigma') \tau^{(\sigma', \sigma)} \epsilon(\sigma', \sigma) D([\sigma' + \sigma]) \\ &= \frac{1}{N^2} \sum_{\sigma'' \in \lambda} \left\{ \sum_{\substack{\sigma, \sigma' \in \lambda \\ [\sigma' + \sigma] = \sigma''}} S(\sigma) S(\sigma') \tau^{(\sigma', \sigma)} \epsilon(\sigma', \sigma) \right\} D(\sigma''). \end{aligned} \tag{45}$$

From the subgroup property of λ : $[\sigma' + \sigma]$ goes over all of λ ; for given $\sigma, \sigma'' \in \lambda, \sigma' \in \lambda$ is unique. The factor $\tau^{(\sigma', \sigma)} = \pm 1$. So in the last expression, for each σ'' , $\{\dots\}$ has exactly N terms, each ± 1 . So, since $\text{Tr}(P_\lambda) = 1$,

$P_\lambda^2 = P_\lambda \iff P_\lambda$ is a rank-1 projection operator

$$\begin{aligned} &\iff \forall \sigma'' \in \lambda, \frac{1}{N} \sum_{\substack{\sigma, \sigma' \in \lambda \\ [\sigma' + \sigma] = \sigma''}} S(\sigma) S(\sigma') \tau^{(\sigma', \sigma)} \epsilon(\sigma', \sigma) = S(\sigma'') \\ &\iff S(\sigma) S(\sigma') = \tau^{(\sigma', \sigma)} \epsilon(\sigma', \sigma) S([\sigma + \sigma']), \quad \forall \sigma', \sigma \in \lambda. \end{aligned} \tag{46}$$

If this is obeyed, it implies that $\{S(\sigma) D(\sigma), \sigma \in \lambda\}$ give a true N -dimensional UR of λ . We have to examine: can these conditions be imposed consistently for all isotropic lines λ ? If yes, to what extent are $S(\sigma)$ then determined? These are the questions we examine next.

5. The N odd case

If N is odd so is N^2 and hence from the group structure we have

$$\sigma \in \Gamma_0 \text{ or } \lambda \implies \exists \text{ unique } \sigma' \in \Gamma_0 \text{ or } \lambda \text{ such that } \sigma = [2\sigma'],$$

$$\text{i.e. any } \sigma = (q, p) = ([2q'], [2p']), \text{ unique } q', p' \in \mathbb{Z}_N. \tag{47}$$

$$q \text{ or } p \text{ even, } \leq N - 1 : q' = q/2 \leq (N - 1)/2, p' = p/2 \leq (N - 1)/2$$

$$q \text{ or } p \text{ odd, } \leq N - 2 : q' = (q + N)/2 \geq (N + 1)/2, p' = (p + N)/2 \geq (N + 1)/2.$$

In relation (46), setting $\sigma = \sigma'$ to get

$$S([2\sigma']) = \epsilon(\sigma', \sigma'), \tag{48}$$

and looking at q/p even/odd we find that $S(q, p)$ are unambiguously given by

$$S(q, p) = (-1)^{qp}. \tag{49}$$

Does this obey condition (46) for all λ ? We find that this is indeed so. The expression $(-1)^{qp}$ was found by taking $\sigma = \sigma'$ in (46). Now we put this into that equation with σ and σ' independent and ask if it is obeyed, i.e. whether or not

$$\epsilon(\sigma', \sigma) = S(\sigma')S(\sigma)S([\sigma' + \sigma])\tau^{(\sigma', \sigma)} \tag{50}$$

holds for all $\sigma', \sigma \in \lambda$. A key observation here is that since N is odd

$$\langle \sigma', \sigma \rangle = 0 \text{ mod } N \implies \langle \sigma', \sigma \rangle = mN \implies$$

$$\tau^{(\sigma', \sigma)} = (-1)^m \implies \tau^{(\sigma', \sigma)} = (-1)^{mN} = (-1)^{\langle \sigma', \sigma \rangle}. \tag{51}$$

So the question now is whether

$$\epsilon(\sigma', \sigma) = (-1)^{q'p' + qp + [q+q'] [p+p'] + q'p - p'q}. \tag{52}$$

We now check the exponent on the rhs in various cases:

$q + q'$	$p + p'$	exponent	rhs
$\leq N - 1$	$\leq N - 1$	$2q'p$	+1
$\leq N - 1$	$\geq N$	$N(q' + q)$	$(-1)^{q'+q}$
$\geq N$	$\leq N - 1$	$N(p' + p)$	$(-1)^{p'+p}$
$\geq N$	$\geq N$	$N^2 + N(q' + q + p' + p)$	$(-1)^{q'+q+p'+p+N}$

$$\tag{53}$$

So, comparing with $\epsilon(\sigma', \sigma)$, we find that (52) holds.

Thus, in the N odd case the marginal conditions for all isotropic lines can be satisfied; all $S(\sigma)$ are determined as above.

This unique solution is related to the Fourier matrix (actually parity matrix) results. The Fourier operator F on \mathcal{H} has these actions and properties:

$$F|q\rangle = |q\rangle, \quad F|p\rangle = |[-p]\rangle, \quad F^\dagger F = FF^\dagger = F^4 = 1; \tag{54}$$

$$F^2|q\rangle = |[-q]\rangle, \quad F^2|p\rangle = |[-p]\rangle. \tag{55}$$

So $F^2 = P =$ parity operator, which is what we need. From relation (21)

$$F|p\rangle\langle q| = \frac{1}{N^{3/2}} \sum_{\sigma' \in \Gamma_0} \omega^{(\sigma, \sigma') + qp} \tau^{-q'p'} D(\sigma'). \tag{56}$$

Set $p = q$ and sum to get

$$F = \frac{1}{N^{3/2}} \sum_{\sigma' \in \Gamma_0} \left\{ \sum_{q \in \mathbb{Z}_N} \omega^{q^2 + q(q' - p')} \right\} \tau^{-q'p'} D(\sigma'). \tag{57}$$

Next for $P = F^2$, by calculating in the $|q'\rangle$ basis

$$\text{Tr}(PD(\sigma)^\dagger) = \tau^{-qP} \sum_{q' \in \mathbb{Z}_N} \omega^{-Pq'} \delta_{[q+2q'],0}, \tag{58}$$

and in the $|p'\rangle$ basis

$$\text{Tr}(PD(\sigma)^\dagger) = \tau^{qP} \sum_{p' \in \mathbb{Z}_N} \omega^{qP'} \delta_{[p+2p'],0}, \tag{59}$$

which are necessarily equal. Using the first form, for N odd, the Kronecker delta gives

$$q = \text{even} = 0, 2, 4, \dots, N - 3, N - 1 : q' = [N - q/2] \tag{60}$$

$$q = \text{odd} = 1, 3, 5, \dots, N - 2 : q' = (N - q)/2 \tag{61}$$

$$\begin{aligned} \text{Tr}(PD(\sigma)^\dagger) &= \begin{cases} q \text{ even: } \tau^{-qP} \omega^{-P[N-q/2]} = 1 \\ q \text{ odd: } \tau^{-qP} \omega^{-P(N-q)/2} = (-1)^P \end{cases} \\ &= (-1)^{qP}. \end{aligned} \tag{62}$$

Hence,

$$P = \frac{1}{N} \sum_{\sigma \in \Gamma_0} (-1)^{qP} D(\sigma). \tag{63}$$

Now

$$\widehat{W}(0, 0) = \frac{1}{N} \sum_{\sigma \in \Gamma_0} S(\sigma) D(\sigma), \tag{64}$$

and hence

$$W(0, 0) = P \iff S(\sigma) = (-1)^{qP}. \tag{65}$$

Thus, for the case of N odd, we see that there is a unique consistent solution for all signs $S(\sigma)$, such that the marginal conditions can be satisfied for all λ 's. The resulting Wigner phase-point operators are the same as those known in the literature and are characterized by the fact that the phase-point operator at the origin is the parity operator as in the continuum case. The existence of a unique square root group element $\sqrt{g(\sigma)}$ for each $g(\sigma)$, guaranteed by N being odd, is adequate for this purpose. In particular it has not been necessary to survey in any sense the set of all isotropic lines λ , and their orbit structure under $SL(2, \mathbb{Z}_N)$ action (see below) etc.

6. The case of N a prime power

Towards handling the case of general N (essentially even N) we may note the following: any N can be uniquely written as the product of powers of (increasing) primes as

$$\begin{aligned} N &= N_1 N_2 \dots N_k = \prod_{j=1}^k N_j \\ N_j &= p_j^{n_j}, p_j = j\text{th prime: } p_1 = 2, p_2 = 3, p_3 = 5, \dots, \\ p_j &= \text{odd } j \geq 2; \quad \text{and } n_j = 0 \text{ or } 1 \text{ or } 2 \dots \end{aligned} \tag{66}$$

If $n_1 = 0$, N is odd and then previous results of section 5 are in hand. We expect something new to arise only when $n_1 \geq 1$.

We consider the case when N is a power of a single prime in the rest of this section, and turn to the general case (66) later in section 7. Simplifying the notation as much as possible for the moment let us write

$$N = p^n, \quad p \text{ prime, } n = 0, 1, 2, \dots \tag{67}$$

(Care will be taken to avoid this prime p being confused with the second entry in the pair (q, p) corresponding to a general point $\sigma \in \Gamma_0 = \mathbb{Z}_N \times \mathbb{Z}_N$.)

6.1. Isotropic lines and $SL(2, \mathbb{Z}_N)$ orbits for $N = p^n$

For the isotropic lines we have the following results [31].

- (1) The total number of isotropic lines λ is

$$\mathcal{N} = (p^{n+1} - 1)/(p - 1). \tag{68}$$

- (2) The number $\mathcal{N}(\sigma)$ of isotropic lines passing through a point $\sigma \in \Gamma_0 = \mathbb{Z}_N \times \mathbb{Z}_N$ is computed as follows. Any $a \in \mathbb{Z}_N$ can be uniquely written as

$$\begin{aligned} a &= a_0 + a_1 p + a_2 p^2 + \dots + a_{n-1} p^{n-1}, \\ a_j &\in \{0, 1, \dots, p - 1\}, \quad j = 0, 1, \dots, n - 1. \end{aligned} \tag{69}$$

The ‘ p -valuation of a ’ is then the smallest j for which a_j is non-zero:

$$\begin{aligned} v(a) &= p\text{-valuation of } a \\ &= j \text{ such that } a_0 = a_1 = \dots = a_{j-1} = 0, a_j \geq 1. \end{aligned} \tag{70}$$

This definition is unambiguous for $a \geq 1$; in particular, we have

$$\begin{aligned} v(1) &= v(2) = \dots = v(p - 1) = 0; \\ v(p) &= 1, \dots; v(p^2) = 2 \dots; \dots; v(p^{n-1}) = n - 1; \\ v(N - 1) &= v(p^n - 1) = 0 \text{ as } p^n - 1 = (p - 1)(1 + p + p^2 + \dots + p^{n-1}). \end{aligned} \tag{71}$$

We supplement these with the convention

$$v(0) = n \tag{72}$$

based on $p^n = 0 \pmod N$. For $\sigma = (q', p') \in \mathbb{Z}_n \times \mathbb{Z}_N$ we define the p -valuation by

$$v(\sigma) = (v(q'), v(p'))_{<}, \quad q', p' \in \mathbb{Z}_N. \tag{73}$$

Thus, for instance

$$v((0, 0)) = n; v((1, p')) = v((q', 1)) = v((N - 1, p')) = v((q', N - 1)) = 0. \tag{74}$$

Then the number of isotropic lines passing through $\sigma \in \mathbb{Z}_N \times \mathbb{Z}_N$ is

$$\mathcal{N}(\sigma) = \frac{(p^{v(\sigma)+1} - 1)}{(p - 1)}. \tag{75}$$

Comparing with (68) we see that $\mathcal{N} = \mathcal{N}((0, 0))$: this is consistent with the condition that any isotropic line λ must contain $(0, 0)$. For $\sigma = (1 \text{ or } N - 1, p')$, $(q', 1 \text{ or } N - 1)$ we have $v(\sigma) = 0$, $\mathcal{N}(\sigma) = 1$, so only one isotropic line passes through each of these points.

We now turn to the group $SL(2, \mathbb{Z}_N)$:

$$SL(2, \mathbb{Z}_N) = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z}_N; ad - bc = 1 \pmod N \right\}, \quad (76)$$

and its action on Γ_0 and on the family of isotropic lines thereof. For the case at hand namely $N = p^n$ the order $|SL(2, \mathbb{Z}_N)|$ is given by

$$|SL(2, \mathbb{Z}_N)| = p^{3n-2}(p^2 - 1). \quad (77)$$

It acts on the points and isotropic lines in Γ_0 as follows:

$$\begin{aligned} A \in SL(2, \mathbb{Z}_N) : \sigma = (q', p') \in \Gamma_0 &\rightarrow \sigma' = (aq' + bp', cq' + dp') \in \Gamma_0, \\ &: \lambda = \{(q', p')\} \rightarrow \lambda' = \{(aq' + bp', cq' + dp')\}. \end{aligned} \quad (78)$$

From the latter action one finds the following [31].

- (1) The \mathcal{N} isotropic lines divide themselves into $1 + [n/2]$ orbits under $SL(2, \mathbb{Z}_N)$ action, where $[n/2]$ is the integer part of $n/2$. They are denoted by $O_k(p^n)$, $k = 0, 1, \dots, [n/2]$. For $k < n/2$ the orbit contains

$$\mathcal{N}(O_k) = (p + 1)p^{n-2k-1} \quad (79)$$

isotropic lines, while for $k = n/2$ in case n is even we have

$$\mathcal{N}(O_{n/2}) = 1. \quad (80)$$

One can easily check in both cases that

$$\sum_{k=0,1,\dots}^{[n/2]} \mathcal{N}(O_k) = \mathcal{N}. \quad (81)$$

The largest orbit corresponds to $k = 0$ and contains $(p + 1)p^{n-1}$ isotropic lines.

- (2) Only the largest orbit $O_0(p^n)$ has the property that it covers all points in $\Gamma_0 = \mathbb{Z}_N \times \mathbb{Z}_N$. The $(p + 1)p^{n-1}$ isotropic lines in this orbit are all generated by single generators of order N which may be taken to be $(1, p')$ for $p' \in \{0, 1, \dots, N - 1\}$ and $(q', 1)$ for $q' \in \{0, p, 2p, 3p, \dots, (p^{n-1} - 1)p\}$.

6.2. Isotropic lines in the 2^n case

Now we specialize to the case $N = 2^n$, as otherwise N is odd and then the comprehensive results of section 5 are available. Thus, in a sense, this is the most important remaining case. Specializing the above statements (and further quoting from [31]) we now have the following.

- (1) The total number of isotropic lines λ is

$$\mathcal{N} = 2^{n+1} - 1 = 2N - 1. \quad (82)$$

- (2) If σ is of the form $(2j, 2k)$, then from (75), as $v(\sigma) \geq 1$, the number of λ 's passing through it is

$$\mathcal{N}(\sigma = (2j, 2k)) \geq 3. \quad (83)$$

If σ is of any of the other three forms $(2j, 2k + 1)$, $(2j + 1, 2k)$ or $(2j + 1, 2k + 1)$, then as $v(\sigma) = 0$ the number of λ 's passing through it is

$$\mathcal{N}(\sigma \neq (2j, 2k)) = 1. \quad (84)$$

- (3) The λ 's separate into two types:
 - type (a): $3N/2$ in number, generated by single generators, and comprising a single (the largest) orbit $O_0(2^n)$,
 - type (b): $N/2 - 1$ in number, involving two generators of orders 2^r and 2^s with both r, s non-zero and $r + s = n$, and comprising all the remaining orbits $O_k(2^n), k = 1, 2, \dots, [n/2]$.
- (4) Every phase point σ lies on (one or more) λ 's of type (a). The λ 's of type (b) cover all the even phase points $(2j, 2k)$ only.
- (5) The λ 's of type (a) separate further into two subtypes:
 - type (a1) containing N λ 's generated by $(1, p_0)$ for $p_0 = 0, 1, 2, \dots, N - 1$;
 - type (a2) containing $N/2$ λ 's generated by $(q_0, 1)$ for $q_0 = 0, 2, 4, \dots, N - 2$.
 Therefore, each σ of any of the three types other than $(2j, 2k)$ lies on a unique λ of type (a) according to the pattern:

$$\begin{aligned} \sigma &= (2j + 1, 2k \text{ or } 2k + 1) && \text{type (a2)} \\ \sigma &= (2j, 2k + 1) && \text{type (a2)}. \end{aligned} \tag{85}$$

6.3. Marginal property for isotropic lines in $\mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n}$

The condition that the average of the phase-point operators along an isotropic line λ is a one-dimensional projector is given in (46). The other essential conditions on the signs $S(\sigma)$ are the reflection symmetry (27) and the standard marginal conditions (33), (34). We know from (39) that $\sigma \in \lambda$ implies $[N - \sigma] \in \lambda$ as well. Applying (46) to such pairs of points on λ 's of type (a), and remembering that any σ lies on such a λ , we find that the property (27) follows. (For λ 's of type (b) this is only partially true as they cover only the even phase points $(2j, 2k)$.) Thus, we begin by imposing only the requirements (33), (34), (46) on $S(\sigma)$, for all λ 's of type (a).

We see from (16), N being even, that

$$\epsilon(\sigma, \sigma) = 1. \tag{86}$$

Therefore, setting $\sigma' = \sigma$ in (46) leads to

$$S((2j, 2k)) = 1. \tag{87}$$

This leaves $S((2j + 1, 2k \text{ or } 2k + 1))$ and $S((2j, 2k + 1))$ to be analysed. Each σ of the former type is on a unique type (a1) λ , while each σ of the latter type is on a unique type (a2) λ . We apply (46) in these cases, choosing $\sigma = \sigma_0 = (1, p_0)$ or $(q_0, 1)$ and $\sigma' = [2j\sigma_0]$ or $[2k\sigma_0]$ respectively, thus reaching all points σ other than $(2j, 2k)$, and relating S at such points to $S(\sigma_0)$:

$$\begin{aligned} S((2j + 1, [(2j + 1)p_0])) &= (-1)^{(2jp_0 - [2jp_0])/N} \cdot S((1, p_0)) \\ &\times \begin{cases} 1 & \text{if } p_0 + [2jp_0] \leq N - 1, \\ -1 & \text{if } p_0 + [2jp_0] \geq N; \end{cases} \end{aligned} \tag{88}$$

$$\begin{aligned} S([(2k + 1)q_0], 2k + 1) &= (-1)^{(2kq_0 - [2kq_0])/N} \cdot S((q_0, 1)) \\ &\times \begin{cases} 1 & \text{if } q_0 + [2kq_0] \leq N - 1, \\ -1 & \text{if } q_0 + [2kq_0] \geq N. \end{cases} \end{aligned} \tag{89}$$

In the former relation, the choice of $2k$ or $2k + 1$ determines p_0 uniquely; in the latter, that of $2j$ determines q_0 uniquely. For $q_0 = 0$, equation (33) determines $S((0, 1)) = 1$; for $p_0 = 0$, equation (33) determines $S((1, 0)) = 1$. The remaining $3N/2 - 2$ undetermined signs are $S((q_0, 1)), q_0 = 2, 4, \dots, N - 2$, and $S((1, p_0)), p_0 = 1, 2, \dots, N - 1$.

These conditions may equivalently be written as

$$S((2j+1, [(2j+1)p_0])) = S((1, p_0)) \times \begin{cases} 1 & \text{if } ((2j+1)p_0 - [(2j+1)p_0])/N \text{ is even} \\ -1 & \text{if } ((2j+1)p_0 - [(2j+1)p_0])/N \text{ is odd} \end{cases} \quad (90)$$

$$S([(2k+1)q_0, (2k+1)]) = S((q_0, 1)) \times \begin{cases} 1 & \text{if } ((2k+1)q_0 - [(2k+1)p_0])/N \text{ is even} \\ -1 & \text{if } ((2k+1)q_0 - [(2k+1)p_0])/N \text{ is odd} \end{cases} \quad (91)$$

For $n = 1, 2$, when $N = 2$ and 4 respectively, we indicate below the free signs at the corresponding locations in the respective phase space:

$$\begin{matrix} 1 & S(1, 1) \\ 1 & 1 \end{matrix} \quad (92)$$

$$\begin{matrix} 1 & S(1, 3) & -S(1, 2) & S(1, 1) \\ 1 & S(2, 1) & 1 & -S(2, 1) \\ 1 & S(1, 1) & S(1, 2) & S(1, 3) \\ 1 & 1 & 1 & 1. \end{matrix} \quad (93)$$

In summary, the marginal conditions can be consistently imposed on all isotropic lines of type (a) comprising the largest orbit but leaving $3 \times 2^{n-1} - 2$ of the $S(\sigma)$ unfixed.

We next see by low-dimensional examples that these conditions cannot be consistently extended to include isotropic lines of type (b). For $n = 1, N = 2$, there are no isotropic lines of type (b). For $n = 2, N = 4$, there is one isotropic line of type (b), generated by $(2, 0)$ and $(0, 2)$. Condition (46) when applied to this isotropic line gives $S(2, 2) = -1$ conflicting with $S(2, 2) = 1$ obtained from isotropic lines of type (a). For $n = 3, N = 8$, there are three isotropic lines of type (b) generated respectively by $\{(2, 0), (0, 4)\}$, $\{(0, 2), (0, 4)\}$, $\{(2, 2), (0, 4)\}$. Again, as for $n = 2, N = 4$ one finds that the results of (46) for isotropic lines of type (b) conflict with those for isotropic lines of type (a)—one cannot impose marginal property on all isotropic lines consistently.

7. Isotropic lines and orbits in the general case

Turning now to the case of a general N we note that the ring \mathbb{Z}_N can be factored as

$$\mathbb{Z}_N = \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \times \dots \times \mathbb{Z}_{N_k}. \quad (94)$$

The explicit correspondence between elements of \mathbb{Z}_N and those of the rings \mathbb{Z}_{N_j} is provided by the Chinese remainder theorem which tells us that an element $q \in \mathbb{Z}_N$ can be uniquely decomposed as

$$q = \sum_{j=1}^k q_j \cdot v_j \cdot \mu_j, \quad (95)$$

where $q_j = [q \bmod N_j] \in \mathbb{Z}_{N_j}$, $v_j = N/N_j$ and μ_j denotes the (multiplicative) inverse of v_j in \mathbb{Z}_{N_j} . Thus, each element $q \in \mathbb{Z}_N$ can be uniquely represented as an array

$$q \longleftrightarrow \{q_1, q_2, \dots, q_k\}, \quad q_i \in \mathbb{Z}_{N_i}. \quad (96)$$

In particular the elements 0 and 1 are represented by

$$0 \longleftrightarrow \{0, 0, \dots, 0\}; \quad 1 \longleftrightarrow \{1, 1, \dots, 1\}. \quad (97)$$

Further, this correspondence has the good property that

$$\begin{aligned} q + q' &\longleftrightarrow \{q_1 + q'_1, q_2 + q'_2, \dots, q_k + q'_k\}, \quad q_i \text{ and } q'_i \in \mathbb{Z}_{N_i}, \\ qq' &\longleftrightarrow \{q_1q'_1, q_2q'_2, \dots, q_kq'_k\}, \quad q_i \text{ and } q'_i \in \mathbb{Z}_{N_i}. \end{aligned} \quad (98)$$

In view of this and the properties (97) and (99) we have the following results.

- A point $\sigma \in \mathbb{Z}_N \times \mathbb{Z}_N$ can be represented as

$$\sigma \longleftrightarrow \{\sigma_1, \sigma_2, \dots, \sigma_k\}, \quad \sigma_i \in \mathbb{Z}_{N_i} \times \mathbb{Z}_{N_i}. \quad (99)$$

- The symplectic product $\langle \sigma, \sigma' \rangle$ vanishes if and only if each of the components $\langle \sigma_i, \sigma'_i \rangle$ vanishes.
- The group $SL(2, \mathbb{Z}_N)$ also factorizes as

$$SL(2, \mathbb{Z}_N) = SL(2, \mathbb{Z}_{N_1}) \times SL(2, \mathbb{Z}_{N_2}) \times SL(2, \mathbb{Z}_{N_k}). \quad (100)$$

This can easily be seen [34] by considering the case $N = N_1 N_2$ and verifying that any matrix $A \in SL(2, \mathbb{Z}_N)$:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}; \quad ab - cd = 1; \quad a, b, c, d \in \mathbb{Z}_n; \quad (101)$$

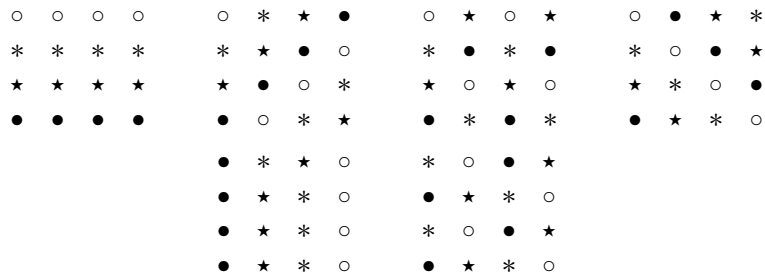
can be decomposed as $A = A_1 A_2$ where

$$A_1 = \begin{pmatrix} (a_1, 1) & (b_1, 0) \\ (c_1, 0) & (d_1, 1) \end{pmatrix} \in SL(2, \mathbb{Z}_{N_1}); \quad A_2 = \begin{pmatrix} (1, a_2) & (0, b_2) \\ (0, c_2) & (1, d_2) \end{pmatrix} \in SL(2, \mathbb{Z}_{N_2}). \quad (102)$$

From these considerations it is evident that the isotropic lines in $\sigma \in \mathbb{Z}_N \times \mathbb{Z}_N$ and $SL(2, \mathbb{Z}_N)$ action are completely determined by those in each of the factors $\mathbb{Z}_{N_j} \times \mathbb{Z}_{N_j}$.

8. Tomographic reconstruction for $N = 2^n$

From the discussion towards the end of section 6 it is evident that for the case when $N = 2^n$ we can only insist on marginal property restricted to the isotropic lines of type (a) constituting the largest orbit under $SL(2, \mathbb{Z}_N)$ action. For each choice for the free signs, we can associate with each such isotropic line a rank-1 projector P_λ . Each isotropic line generates $N - 1$ other lines ‘parallel’ to it obtained, for instance, by shifting the points on it by an amount $(0, i)$ in the case of isotropic lines of type (a1) and by an amount $(i, 0)$ in the case of isotropic lines of type (a2) with i taking values $1, 2, \dots, N - 1$. We denote by $(\lambda, i), i = 0, 1, \dots, N - 1$, the set of N lines consisting of the isotropic line λ and the lines parallel to it. Each such set (λ, i) furnishes a striation of Γ_0 in the sense of Wootters *et al* [7]. Thus, for $N = 4$, there are six isotropic lines of type (a)—four of type (a1) and two of type (a2). The associated striations of Γ_0 are displayed below where the lines parallel to the isotropic line (indicated by \bullet) obtained by shifts, upwards for isotropic lines of type (a1) and rightwards for an isotropic line of type (a2), by an amount $i = 1, 2, 3$ are indicated by $\star, *, \circ$, respectively.



Starting from the projector P_λ corresponding to the isotropic line λ , we assign projectors $P_{(\lambda,i)}$ to the other lines in that striation obtained by conjugate action of the appropriate displacement operators on P_λ :

$$P_{(\lambda,i)} = \begin{cases} D(0, i) P_\lambda D^\dagger(0, i) & \text{if } \lambda \text{ is of type (a1)} \\ D(i, 0) P_\lambda D^\dagger(i, 0) & \text{if } \lambda \text{ is of type (a2)}. \end{cases} \quad (103)$$

From the definition of $P_{(\lambda,i)}$ it immediately follows that

$$\sum_{i=0}^N P_{(\lambda,i)} = \mathbb{I}. \tag{104}$$

The collection of projectors $\{P_{(\lambda,i)}\}$ associated with all striations then provides us with a quantum net, to borrow a term introduced by Wootters *et al* [7] in the context of Wigner distributions on finite fields. As there are $3N/2$ isotropic lines of type (a), this collection contains $3N^2/2$ projectors. Using the results obtained earlier, after some algebra, we can compute the inner product of any two projectors in this set and find

$$\text{Tr}[P_{(\lambda,i)}P_{(\lambda',j)}] = \frac{1}{N} \sum_{\sigma=(q,p) \in \lambda \cap \lambda'} \begin{cases} \omega^{(i-j)q} & \text{if both } \lambda \text{ and } \lambda' \text{ of type (a1)} \\ \omega^{(iq+jp)} & \text{if } \lambda \text{ of type (a1) and } \lambda' \text{ of type (a2)} \\ \omega^{-(ip+jq)} & \text{if } \lambda \text{ of type (a2) and } \lambda' \text{ of type (a1)} \\ \omega^{-(i-j)p} & \text{if both } \lambda \text{ and } \lambda' \text{ of type (a2)}. \end{cases} \tag{105}$$

Two results follow immediately. Putting $\lambda = \lambda'$ in the above equation we obtain

$$\text{Tr}[P_{(\lambda,i)}P_{(\lambda,j)}] = \delta_{ij}, \tag{106}$$

i.e. the N projectors associated with each striation constitute a trace orthonormal set.

Further, as isotropic lines of type (a1) and those of type (a2) have no points in common except $(0, 0)$, we find that if λ and λ' are of different types,

$$\text{Tr}[P_{(\lambda,i)}P_{(\lambda',j)}] = \frac{1}{N}. \tag{107}$$

In other words the projectors associated with isotropic lines of different types are *mutually unbiased*. If the two isotropic lines are of the same type, using the properties of isotropic lines and those of the points that lie on them as discussed earlier, one finds that for $N = 2^n$ two such lines can have 2^m points in common where $m = 0, 1, \dots, n$ and in that case one has

$$\text{Tr}[P_{(\lambda,i)}P_{(\lambda',j)}] = \frac{1}{N} \begin{cases} 2^m & \text{if } |i - j| = 0 \pmod{2^m}, \\ 0 & \text{otherwise.} \end{cases} \tag{108}$$

In fact, the expressions for $\text{Tr}[P_{(\lambda,i)}P_{(\lambda',j)}]$ above can be unified into a single statement

$$\text{Tr}[P_{(\lambda,i)}P_{(\lambda',j)}] = \frac{1}{N} \times [\text{number of points common to the lines } (\lambda, i) \text{ and } (\lambda', j)]. \tag{109}$$

If one regards the number of points common to two lines (λ, i) and (λ', j) divided by N as the ‘phase-space overlap’ between the two lines, then the equation above tells us that the ‘phase-space overlap’ between two lines exactly equals the quantum mechanical overlap between the associated projectors. With these results at hand, we can explicitly write down the matrices of overlaps $\text{Tr}[P_{(\lambda,i)}P_{(\lambda',j)}]$ for any λ and λ' . Thus, for $N = 4$ they are

$$\frac{1}{4} \begin{pmatrix} I_1 & I_1 & I_1 & I_1 \\ I_1 & I_1 & I_1 & I_1 \\ I_1 & I_1 & I_1 & I_1 \\ I_1 & I_1 & I_1 & I_1 \end{pmatrix}; \quad \frac{2}{4} \begin{pmatrix} I_2 & I_2 \\ I_2 & I_2 \end{pmatrix}; \quad \frac{4}{4} I_4 \tag{110}$$

depending on whether λ and λ' have 1, 2 or 4 points in common. Here, I_r denotes the r -dimensional unit matrix. Similar pattern holds for any $N = 2^n$.

From these N projectors associated with each striation we can construct $N - 1$ traceless operators $T_{(\lambda,i)} = P_{(\lambda,i)} - I_N/N$. We therefore have a collection of $3N(N - 1)/2$ traceless hermitian operators. Given a density operator ρ for an N -state system, the operator $\rho - \mathbb{I}/N$ belongs to the $(N^2 - 1)$ -dimensional real Hilbert space of $N \times N$ traceless Hermitian operators.

The question concerning the tomographic reconstruction of ρ then reduces to the question whether or not the collection of the T 's above spans the $(N^2 - 1)$ -dimensional real vector space of traceless hermitian operators. This can be checked by examining the rank of the Gram matrix associated with the T 's. This is easily done using the results above. For $N = 2, 4$ we have explicitly checked that the corresponding Gram matrices indeed have ranks 3 and 15, respectively. Thus, it would seem that even with restricted marginal property the construction developed here permits a tomographic reconstruction of the state of an N -level system though in a non-optimal fashion—we have $(N - 1)(N - 2)/2$ more T 's than the $N^2 - 1$ required.

9. Concluding remarks

Following the Dirac-inspired approach developed by us earlier [26], we show how to set up Wigner distributions for finite-dimensional quantum systems working entirely with an $N \times N$ phase-space lattice regardless of whether N is even or odd. This is in contrast with the existing formalisms where, for even N , it was found necessary to introduce a great deal of redundancy in the description to arrive at a satisfactory definition of a Wigner distribution. These include those where one is required to double the number of coordinates from N to $2N$ and work with a $2N \times 2N$ phase-space grid. The approach adopted here, based on finding the square root of a certain matrix kernel, introduces undetermined signs, one at each phase-space point of the $N \times N$ phase-space grid and the strategy then is to derive conditions on them by imposing marginal property on all the isotropic lines. To 'solve' the relations thus obtained one naturally requires a detailed knowledge of the isotropic lines on an $N \times N$ grid and their properties. We summarize relevant details based on the results of Albouy [31] obtained in the context of more general Lagrangian subspaces and use them towards fixing or relating the undetermined signs. We find that the N even and odd cases differ from each other in two important respects. In the N odd case one can consistently impose the marginal property on all isotropic lines and the Wigner distribution thus obtained is uniquely defined. In fact, as it turns out, detailed knowledge of the structure of the isotropic lines in this case is unnecessary and the only property that one ever uses is the fact that the 'midpoint' of every point on an isotropic line and the origin also lies on that isotropic line. In the even case, on the other hand, this is not so and a detailed knowledge of the structure of the isotropic lines is indispensable. We find that here the marginal property cannot be imposed consistently on all the isotropic lines but rather on subsets thereof consisting of individual orbits under the $SL(2, \mathbb{Z}_N)$ group action. We present results concerning the structure of these orbits and find that from the point of tomography only the largest of such orbits is of interest as it is the only one which covers all the phase-space points. Confining ourselves only to such orbits and therefore to a restricted marginal property, we are led to a family of Wigner distributions, $3 \cdot 2^{n-1} - 2$ in number where n is the exponent of 2 in the decomposition of N into prime factors. As a result, for instance, for $N = 2, 4, 8, 16$ there are 2, 2^4 , 2^{10} , 2^{22} different possible definitions of Wigner distributions.

As a curiosity, based on the work in [35], we have also examined the dependence of the eigenvalues of the phase-point operators for $N = 2, 4, 8$, as a function of the signs that remain free. (For this purpose it is sufficient to observe the eigenvalues of $\widehat{W}(0, 0)$.) We find the following.

For $N = 2$ there is only one free sign, $S(1, 1)$, and the spectrum of $\widehat{W}(0, 0)$ is the same for $S(1, 1) = \pm 1$.

For $N = 4$ there are three distinct spectra for $\widehat{W}(0, 0)$ depending on the values of the four free signs, $S(1, 1) \equiv a$, $S(1, 2) \equiv b$, $S(1, 3) \equiv c$, $S(2, 1) \equiv d$. They are

- $((1 + \sqrt{6})/2, (1 - \sqrt{6})/2, -1/2, 1/2)$ corresponding to $a = c, b = -d$ and $a = -c, b = d$;
- $((1 + 2\sqrt{2})/2, -1/2, (1 - \sqrt{2})/2, (1 - \sqrt{2})/2)$ corresponding to $a = c = 1, b = d = 1, a = c = -1, b = d = -1$ and $a = -c = 1, b = -d$;
- $((1 + \sqrt{2})/2, (1 + \sqrt{2})/2, (1 - 2\sqrt{2})/2, -1/2)$ corresponding to $a = c = 1, b = d = -1, a = c = -1, b = d = 1$ and $a = -c = -1, b = -d$.

For $N = 8$, one has 4 and for $N = 16$, one has 15 distinct spectra. Thus, although the number of different Wigner distributions based on choices for the signs for $N = 2, 2^2, 2^3, 2^4$ is $2, 2^4, 2^{10}, 2^{22}$, those which have distinct spectra are only 1, 3, 4, 15 in number. The question as to what bring about this enormous reduction is under investigation. Further, it would be interesting to see if the square root idea developed here works in the case when the coordinates take values in a finite field [7, 11] and to see how it relates to Wigner distributions in the more general setting based on the theory of frames employed in the work of Ferrie and Emerson [29].

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Appendix A

In this appendix, we give the matrix elements in the coordinate basis for (a) position and momentum operators \hat{q} and \hat{p} , (b) displacement operators $D(q, p)$, (c) phase-point operators $\hat{W}(q, p)$ and (d) projectors $P_{(\lambda, i)}$ associated with the line (λ, i) . These matrices are explicitly stated for $N = 2$.

From the definitions given in sections 2–4, it follows that in the coordinate basis

$$[\hat{q}]_{\ell k} = \ell \delta_{\ell k}, \tag{A.1}$$

$$[\hat{p}]_{\ell k} = \frac{1}{N} \sum_{p=1}^{N-1} p \omega^{p(\ell-k)}, \tag{A.2}$$

$$[D(q, p)]_{\ell k} = \tau^{qp} \omega^{kp} \delta_{\ell, [k+q]}, \tag{A.3}$$

$$[\hat{W}(q, p)]_{\ell, k} = \frac{1}{N} \sum_{(q', p') \in \Gamma_0} \omega^{(pq' - qp')} S(q', p') [D(q', p')]_{\ell k}, \tag{A.4}$$

$$[P_{\lambda}]_{\ell k} = \frac{1}{N} \sum_{(q', p') \in \lambda} S(q', p') [D(q', p')]_{\ell k}, \tag{A.5}$$

where $\omega = e^{2\pi i/N}$ and $\tau = e^{\pi i/N}$ and ℓ, k take values $0, 1, \dots, N - 1$. The matrices for $P_{(\lambda, i)}$ can then be obtained by applying the matrices for appropriate displacement operators as

in (103). For $N = 2$ the matrices for the operators listed above turn out to be as follows.

(a) *Position and momentum operators*

$$\hat{q} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}; \quad \hat{p} = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}. \quad (\text{A.6})$$

(b) *Displacement operators*

$$D(0, 0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad D(0, 1) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \quad (\text{A.7})$$

$$D(1, 0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad D(1, 1) = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \quad (\text{A.8})$$

(c) *Phase-point operators*

$$\hat{W}(0, 0) = \frac{1}{2} \begin{pmatrix} 2 & 1 - ia \\ 1 + ia & 0 \end{pmatrix}; \quad \hat{W}(0, 1) = \frac{1}{2} \begin{pmatrix} 2 & -1 + ia \\ -1 - ia & 0 \end{pmatrix}; \quad (\text{A.9})$$

$$\hat{W}(1, 0) = \frac{1}{2} \begin{pmatrix} 0 & 1 + ia \\ 1 - ia & 2 \end{pmatrix}; \quad \hat{W}(1, 1) = \begin{pmatrix} 0 & -1 - ia \\ -1 + ia & 2 \end{pmatrix}. \quad (\text{A.10})$$

Here and below $a \equiv S(1, 1)$ takes values ± 1 . (d) *Projectors* For $N = 2$, there are three isotropic lines (indicated by \bullet): two of type (a1) and one of type (a2). Each isotropic line has one line parallel to it (indicated by \star). The state vectors $|\lambda, i\rangle$ associated with the rank-1 projectors $P_{(\lambda,i)} = |\lambda, i\rangle\langle\lambda, i|$ are displayed below:

$$\begin{array}{l} \star \star; \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \quad \star \bullet; \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ ia \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -ia \end{pmatrix} \right\} \\ \bullet \bullet; \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}. \end{array} \quad (\text{A.11})$$

References

[1] Wigner E P 1932 *Phys. Rev.* **40** 749
 [2] Hillery M, O'Connell R F, Scully M O and Wigner E P 1984 *Phys. Rep.* **106** 121
 [3] Kim Y S and Noz M E 1991 *Phase-Space Picture of Quantum Mechanics* (Singapore: World Scientific)
 Schleich W P 2001 *Quantum Optics in Phase Space* (Weinheim: Wiley-VCH)
 [4] Hannay J H and Berry M V 1980 *Physica D* **1** 26
 [5] Leonhardt U 1996 *Phys. Rev. A* **53** 2998
 Bianucci P, Miguel C, Paz J P and Saraceno M 2005 *Phys. Lett. A* **72** 012309
 [6] Bouzouina A and De Bievre S 1998 *Commun. Math. Phys.* **178** 83
 [7] Wootters W K 1987 *Ann. Phys., NY* **176** 1
 Gibbons K S, Hoffman M J and Wootters W K 2004 *Phys. Rev. A* **70** 062101
 [8] Hakioglu T 1998 *J. Phys. A: Math. Gen.* **31** 6975
 [9] Cohendet O, Combe P, Sirugue M and Sirugue-Collin M 1987 *J. Phys. A: Math. Gen.* **21** 2875
 [10] Rivas A M F and Ozorio de Almeida A M 1999 *Ann. Phys. A* **276** 223
 Rivas A M F, Saraceno M and Ozorio de Almeida A M 2000 *Nonlinearity* **13** 341
 Rivas A M F and Ozorio de Almeida A M 2002 *Nonlinearity* **15** 681
 Rivas A M F 2007 *J. Phys. A: Math. Theor.* **40** 11057
 [11] Vourdas A 2004 *Rep. Prog. Phys.* **67** 267
 Wang L, Al Hadhrami H and Vourdas A 2008 *Eur. Phys. J. D* **49** 265
 [12] Miquel C, Paz J P and Saraceno M 2002 *Phys. Rev. A* **65** 062309
 [13] Muñoz Villegas C A, Chavez Chavez A, Chumakov S, Fofanov Yu and Klimov A B 2003
 arXiv:quant-ph/0307051

- Klimov A B and Muñoz C 2005 *J. Opt. B* **7** S588
Klimov A B, Romero J L, Bjork G and Sanchez-Soto L L 2009 *Ann. Phys., NY* **324** 53
Klimov A B, Muñoz C, Fernandez A and Saavedra C 2009 *Phys. Rev. A* **77** 060303R
- [14] Bjork G, Klimov A B and Sanchez-Soto L L 2008 *Prog. Opt.* **51** 469
Klimov A B and Romero J L 2008 *J. Phys. A: Math. Theor.* **41** 055303
- [15] Galetti D and Toledo Piza A F R 1988 *Physica A* **149** 267
Galetti D and Toledo Piza A F R 1992 *Physica A* **186** 513
Ruzzi M, Galetti D and Machioli M A 2005 *J. Phys. A: Math. Gen.* **38** 6239
Ruzzi M, Galetti D and Machioli M A 2005 *Phys. Rev. A* **72** 042308
- [16] Luis A and Peřina J 1998 *J. Phys. A: Math. Gen.* **31** 1423
[17] Oparný T, Welsh D-G and Bužek V 1996 *Phys. Rev. A* **53** 3822
[18] Vaccaro J A and Pegg D T 1990 *Phys. Rev. A* **41** 5156
[19] Stratonovich R L 1957 *J. Exp. Theor. Phys.* **4** 891
[20] Agarwal G S 1981 *Phys. Rev. A* **24** 2889
[21] Hess S and Weigert S 2000 *Phys. Rev. A* **63** 012105
[22] Gross D 2005 *Diploma Thesis* University of Potsdam, Available online at <http://gross.qipc.org>
Gross D 2006 *J. Math. Phys.* **47** 122107
- [23] Mukunda N, Arvind Chaturvedi S and Simon R 2004 *J. Math. Phys.* **45** 114
Mukunda N, Marmo G, Zampini A, Chaturvedi S and Simon R 2005 *J. Math. Phys.* **46** 012106
- [24] Mukunda N, Chaturvedi S and Simon R 2004 *Phys. Lett. A* **321** 160
[25] Chaturvedi S, Ercolessi E, Marmo G, Morandi G, Mukunda N and Simon R 2006 *Pramana J. Phys.* **65** 981
[26] Chaturvedi S, Ercolessi E, Marmo G, Morandi G, Mukunda N and Simon R 2006 *J. Phys. A: Math. Gen.* **39** 1405
[27] Ibort A, Manko V I, Marmo G, Simoni A and Ventiglia F 2006 *J. Phys. A: Math. Theor.* **42** 155302
[28] Narnhofer H 2006 *J. Phys. A: Math. Gen.* **39** 7051
[29] Ferrie C and Emerson J 2009 *New J. Phys.* **11** 063040
Ferrie C and Emerson J 2008 *J. Phys. A: Math. Theor.* **41** 352001
- [30] Bar-on T 2009 *J. Math. Phys.* **50** 072106
[31] Albouy O 2009 *J. Phys. A: Math. Theor.* **42** 072001
- [32] Zauner G 1999 Quantum designs foundations of a non-commutative theory of designs (in German) *PhD Thesis* University of Vienna, Available online at <http://www.mat.univie.ac.at/neum/papers/physpapers.html>
Renes J M, Blume-Kohout R, Scott A J and Caves C M 2004 *J. Math. Phys.* **45** 2171
Grassl M 2005 *Electron. Notes Discrete Math.* **20** 151
Appleby D M 2005 *J. Math. Phys.* **46** 052107
- [33] Shirokova S A 1968 *Russ. Math. Surv.* **23** 47
Bryant V 1993 *Aspects of Combinatorics: A Wide Ranging Introduction* (Cambridge: Cambridge University Press)
- [34] Athanasiu G G, Floratos E G and Nicolis S 1998 *J. Phys. A: Math. Gen.* **31** L665
[35] Appleby D M, Bengtsson I and Chaturvedi S 2008 *J. Math. Phys.* **49** 012102